Limitation on stabilizing plane waves via time-delay feedback

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(Received 27 April 2001; published 15 October 2001)

Previous work has demonstrated the possibility of stabilizing plane wave solutions of one-dimensional systems using a spatially local form of time-delayed feedback. We show that the natural extension of this method to two-dimensional systems fails due to the presence of torsion-free unstable perturbations. Linear stability analysis of the complex Ginzburg-Landau equation reveals that long wavelength, transverse wave instabilities cannot be suppressed by the method of extended time-delay autosynchronization. The conclusion follows from symmetry considerations and therefore applies to a wide class of models with simple plane wave solutions.

DOI: 10.1103/PhysRevE.64.056206 PACS number(s): 05.45.Gg, 47.27.Rc, 47.52.+j

I. INTRODUCTION

It has been suggested by Ott, Grebogi, and Yorke [1] that by applying small perturbations to a dynamical system, one can convert a chaotic attractor to any of a large number of time periodic motions. One important element required for exploiting this idea is the ability to stabilize an intrinsically unstable periodic orbit (UPO). In systems that are either too fast or too complex to permit the application of standard control techniques, it is sometimes possible to achieve stabilization using time-delay feedback, which has the advantage of not requiring prior knowledge of anything but the period of the desired orbit.

The method of ''time-delay autosynchronization'' (TDAS) was first introduced by Pyragas [2]. It is based on applying feedback proportional to the deviation of the current state of the system from its state one period in the past. Socolar, Sukow, and Gauthier [3] have proposed an extension of the scheme, referred to as ETDAS, which achieves a larger domain of control in parameter space by using a sum of states at integer multiples of the period in the past. The sum takes the form of a geometric series that can be generated experimentally using only a single time-delay element in a feedback loop. It is known that the ETDAS method can be effective for stabilizing simple systems such as a driven nonlinear pendulum [4], and ETDAS has been demonstrated experimentally in high frequency electronic oscillators [5].

We are interested in the possibility of stabilizing spatially extended systems. Bleich and Socolar [6] showed that ETDAS can be used to enlarge the domain of stability of plane waves in the one-dimensional complex Ginzburg-Landau equation (CGLE). The method studied involved the addition of a spatially local feedback term to the CGLE. Here we address the question of whether those results can be extended to higher-dimensional systems.

We find that in two or more dimensions the unstable plane wave solutions of the CGLE cannot be stabilized by spatially local ETDAS. The reason is that there exist unstable perturbations of the plane waves that have purely real Floquet multipliers (no torsion). A theorem first proven by Nakajima [7] precludes the control of such orbits using straight-forward time-delay control. Adding higher-order terms in the CGLE does not alter this conclusion, so long as the new terms do not destroy the relevant symmetries of the system.

In Sec. II we present a proof that ETDAS cannot stabilize certain torsion-free orbits. Section III contains the analysis of the CGLE, emphasizing the existence of torsion-free instabilities. In Sec. IV we discuss the symmetries responsible for the presence of torsion-free modes in the CGLE and show that spatially local ETDAS still fails in the presence of higher-order terms.

II. LIMITS OF ETDAS

In this section we review the result of Nakajima and Ueda showing that ETDAS cannot stabilize certain torsion-free orbits [7,8]. We present their argument (in a slightly modified form) for completeness and to make clear the application to the CGLE problem discussed in Secs. III and IV.

Let a dynamical variable $\mathbf{B}$ be a complex vector quantity with dynamics governed by the equation

$$\partial_t \mathbf{B}(t) = \mathbf{F}(\mathbf{B}(t)), \tag{1}$$

where $\mathbf{F}$ is a given, smooth function. Let $\mathbf{B}_0(t)$ be a solution of Eq. (1) that is periodic with period τ:

$$\mathbf{B}_0(t+\tau) = \mathbf{B}_0(t). \tag{2}$$

The ETDAS method for stabilizing the UPO consists of the addition of a control term based on the difference between system states separated in time by one period τ. The equation governing the controlled system is

$$\partial_t \mathbf{B}(t) = \mathbf{F}(\mathbf{B}(t)) + \gamma \sum_{l=0}^{\infty} R^l \mathbf{u}_l(t;\tau), \tag{3}$$

where

$$\mathbf{u}_l(t;\tau) = \hat{\mathbf{M}}(t)[\mathbf{B}(t-l\tau) - \mathbf{B}(t-l\tau-\tau)]. \tag{4}$$

Here $\gamma \in \mathrm{Re}$ is the gain, $R \in (-1,1)$ is a parameter that determines the relative importance of past differences, and $\hat{\mathbf{M}}(t)$ is a matrix that specifies the linear transformation relating the feedback signal(s) to the measured components of

B. Note that for any R, when the system is on the desired UPO, we have $\mathbf{u}_l=0$ for all l and the control signal vanishes.

To determine the stability of $\mathbf{B}_0(t)$, let $\mathbf{y}(t)=\mathbf{B}(t)-\mathbf{B}_0(t)$, linearize Eq. (3), and reorganize the sum to obtain

$$\dot{\mathbf{y}}(t)=\hat{\mathbf{J}}(t)\mathbf{y}(t)+\gamma\hat{\mathbf{M}}(t)\left[\mathbf{y}(t)+(R-1)\sum_{l=1}^{\infty}R^{l-1}\mathbf{y}(t-l\tau)\right], \tag{5}$$

where $\hat{\mathbf{J}}(t)$ is the Jacobian of $\hat{\mathbf{F}}$ evaluated on the UPO:

$$J_{ij}(t)=\left.\frac{\partial F_i(\mathbf{B})}{\partial B_j}\right|_{\mathbf{B}=\mathbf{B}_0(t)}. \tag{6}$$

Note that the uncontrolled system is described by Eq. (5) with $\gamma=0$. Since $\hat{\mathbf{J}}(t)$ is evaluated on the periodic orbit and is therefore periodic with period τ, and γ is a real constant, standard Floquet theory allows us to write

$$\mathbf{y}(t)=\sum_n e^{s_n t}\mathbf{p}_n(t), \tag{7}$$

with each $\mathbf{p}_n(t)$ a strictly periodic function with period τ:

$$\mathbf{p}_n(t+\tau)=\mathbf{p}_n(t). \tag{8}$$

The stability of $\mathbf{B}_0$ is determined by the values of s_n.

To determine s_n, consider a single mode $\mathbf{p}_n(t)$. Dropping the subscript n, let

$$\mathbf{v}(t)=e^{st}\mathbf{p}(t). \tag{9}$$

Define a time evolution operator $\hat{\mathbf{U}}(t;\gamma)$ such that

$$\mathbf{v}(t)=\hat{\mathbf{U}}(t;\gamma)\mathbf{v}(0). \tag{10}$$

Substituting $\mathbf{v}(t)$ into Eq. (5), we find

$$\dot{\mathbf{v}}(t)=\left[\hat{\mathbf{J}}(t)+\gamma\frac{1-e^{-s\tau}}{1-Re^{-s\tau}}\hat{\mathbf{M}}(t)\right]\mathbf{v}(t). \tag{11}$$

Substituting Eq. (10) into Eq. (11) and formally integrating, we can write $\hat{\mathbf{U}}(t;\gamma)$ in the following way:

$$\hat{\mathbf{U}}(t;\gamma)=\hat{\mathbf{T}}\left[\exp\int_0^t du\left(\hat{\mathbf{J}}(u)+\gamma\frac{1-z}{1-Rz}\hat{\mathbf{M}}(u)\right)\right]\hat{\mathbf{I}}, \tag{12}$$

with $\hat{\mathbf{T}}$ as the time-ordering product operator, $\hat{\mathbf{I}}$ the identity, and z the inverse Floquet multiplier defined as $z\equiv e^{-s\tau}$. Note that the denominator $1-Rz$ is well behaved for any $R\in(-1,1)$ when $z\neq 1/R$. Thus for $|R|<1$ and any $|z|<1$, integrating over any finite time interval yields finite $\hat{\mathbf{U}}(t;\gamma)$. Due to the periodicity of $\mathbf{p}$, we have

$$\mathbf{v}(t+\tau)=e^{s\tau}\mathbf{v}(t), \tag{13}$$

which, together with Eq. (10), implies

$$|\hat{\mathbf{I}}-z\hat{\mathbf{U}}(\tau;\gamma)|=0. \tag{14}$$

Note that $\hat{\mathbf{U}}(\tau;\gamma)$ depends on z, but not on $\mathbf{p}_n(t)$. Thus the z's that are solutions to the above equation determine all the values of s_n.

The following theorem limits the applicability of ETDAS control in cases where a UPO exhibits no torsion. Note that the value of R does not affect the result as long as $R\in(-1,1)$.

Theorem. Consider an UPO of a dynamical system, for which $\hat{\mathbf{U}}(\tau;0)$ has an odd number of real eigenvalues greater than 1, with all other eigenvalues either real and less than 1 or members of complex conjugate pairs. Let $\hat{\mathbf{M}}(t)$ be any τ-periodic (or constant) matrix that enters the definition of $\hat{\mathbf{U}}(\tau;\gamma)$ as shown in Eq. (12). If the eigenvalues of $\hat{\mathbf{U}}(\tau;\gamma)$ are real or come in complex conjugate pairs for all $\gamma\in\mathrm{Re}$ and $z\in(0,1)$, then the UPO cannot be stabilized via ETDAS by any choice of γ.

Proof. Following Nakajima [7], let

$$G_\gamma(z)\equiv|\hat{\mathbf{I}}-z\hat{\mathbf{U}}(\tau;\gamma)|. \tag{15}$$

The stability of the system is determined by the roots of $G_0(z)$ for the uncontrolled system and $G_\gamma(z)$ for the controlled system. The existence of a root with $|z|<1$ implies instability. $G_0(z)$ is just the characteristic polynomial for the inverse eigenvalues of $\hat{\mathbf{U}}(\tau;0)$, which by assumption has an odd number of roots between 0 and 1. We will prove that $G_\gamma(z)$ has at least one root between 0 and 1.

Let ϕ_l be the eigenvalues of $\hat{\mathbf{U}}(\tau;0)$. Writing Eq. (15) for $\gamma=0$ in the basis where $\hat{\mathbf{U}}(\tau;\gamma)$ is diagonal, we have

$$G_0(z)=\prod_{l=1}^{N}(1-z\phi_l). \tag{16}$$

Now from Eq. (12), we also have $G_\gamma(1)=G_0(1)$, so Eq. (16) implies

$$G_\gamma(1)=\prod_{l=1}^{N}(1-\phi_l) \tag{17}$$

for all γ. Since the number of ϕ_l's that are real and greater than unity is odd and other ϕ_l's come in complex conjugate pairs, $G_\gamma(1)$ must be real and negative. On the other hand, from the definition of $G_\gamma(z)$ we see immediately that $G_\gamma(0)=1$ for all γ.

$G_\gamma(z)$ is continuous and has no singularities for $z\in(0,1)$. Moreover, by assumption, the eigenvalues $\hat{\mathbf{U}}(\tau;\gamma)$ are either real or form complex conjugate pairs for all $z\in(0,1)$, so $G_\gamma(z)$ is real for all $z\in(0,1)$. From $G_\gamma(1)<0$ and $G_\gamma(0)=1$ it then follows that $G_\gamma(z)=0$, for some $z\in(0,1)$. Q.E.D.

The following remarks address special cases of the theorem.

Remark 1. In the case were $\mathbf{B}(t)\in\mathrm{Re}^n$, the quantities $\hat{\mathbf{J}}(t)$, $\hat{\mathbf{M}}(t)$, and γ must all be real, which leads to $\hat{\mathbf{U}}(\tau;\gamma)$ being real for any $z\in\mathrm{Re}$ (excepting $z=1/R$, which lies outside the unit circle and hence does not affect the argument). In this case, the condition for the theorem to apply is simply that

$\hat{\mathbf{U}}(\tau;0)$ has an odd number of real eigenvalues greater than 1. (This is the case addressed directly in Refs. [7] and [8].)

Remarks 2. When both $\hat{\mathbf{J}}$ and $\hat{\mathbf{M}}$ are time independent, the requirements of the theorem reduce to $\hat{\mathbf{J}}$ having an odd number of real eigenvalues greater than zero (and all others either real and less than zero or members of complex conjugate pairs) and $\hat{\mathbf{J}}+r\hat{\mathbf{M}}$ having eigenvalues that are real or come in complex conjugate pairs for any $r\in\mathrm{Re}$.

Remark 3. In the case where $\hat{\mathbf{B}}(t)$ is a two component complex vector, and $\hat{\mathbf{J}}$ and $\hat{\mathbf{M}}$ are time independent, the conditions of the theorem are fulfilled if and only if the eigenvalues of $\hat{\mathbf{J}}$ are pure, real, and have opposite signs and the eigenvalues of $\hat{\mathbf{J}}+r\hat{\mathbf{M}}$, for any $r\in\mathrm{Re}$, are purely, real, or a complex conjugate pair. These conditions require only that both the trace and determinant of $\hat{\mathbf{J}}+r\hat{\mathbf{M}}$ be real for all r and that the determinant of $\hat{\mathbf{J}}$ be negative. For future reference we note that in the special case where both $\hat{\mathbf{J}}$ and $\hat{\mathbf{M}}$ take the form

$$\hat{\mathbf{J}}_1=\begin{bmatrix}\alpha & \beta\\ \beta^* & \alpha^*\end{bmatrix}, \tag{18}$$

the theorem applies whenever $[\hat{\mathbf{J}}]<0$.

In the next sections we show that certain spatially extended systems controlled by ETDAS give rise to instabilities of the form covered by the case $\hat{\mathbf{J}}_1$ of Remark 3.

III. THE COMPLEX GINZBURG-LANDAU EQUATION

The CGLE with the simplest form of ETDAS control is

$$\partial_t A(\mathbf{x},t)=\epsilon A+(1+ic_1)\nabla^2 A-(1-ic_3)|A|^2 A + \gamma m\left[A(\mathbf{x},t)-(1-R)\sum_{l=1}^{\infty} R^{l-1}A(\mathbf{x},t-l\tau)\right], \tag{19}$$

where A is a complex valued field, c_1, c_3, and ϵ are real constants characterizing the system [9], and the last term on the right hand side is the ETDAS control term with m a complex number of unit magnitude. A family of solutions to the above equation is the traveling wave given by

$$A_{\mathbf{k}}(\mathbf{x},t)=a_{\mathbf{k}}e^{i(\mathbf{k}\cdot\mathbf{x}-\Omega_{\mathbf{k}}t)} \tag{20}$$

where $a_{\mathbf{k}}=\sqrt{\epsilon-k^2}$ and $\Omega_{\mathbf{k}}=(c_1+c_3)k^2-c_3\epsilon$. In order for the solution to be physically meaningful, the wave number k must be smaller than $\sqrt{\epsilon}$.

We consider a spatially local and homogeneous ETDAS feedback, $A(\mathbf{x},t)-A(\mathbf{x},t-\tau)$, that has been shown to be effective for the case of one spatial dimension [6]. We will show that in two or more dimensions, if $c_1c_3>1$, there is no choice of γm that stabilizes an unstable plane wave.

We use standard techniques to analyze the stability of the plane wave $A_{\mathbf{k}}(\mathbf{x},t)$. We write $A=(A_{\mathbf{k}}+B)$, and expand B in Fourier modes:

$$B=\sum_{\mathbf{q}} \mu_{\mathbf{q}}(t)e^{i\mathbf{q}\cdot\mathbf{x}}. \tag{21}$$

Substituting the perturbed solution into the CGLE, linearizing in $\mu_{\mathbf{q}}$, canceling a common factor $A_{\mathbf{k}}$, and equating coefficients of $\exp(i\mathbf{q}\cdot\mathbf{x})$ in the sums, we obtain a set of coupled first-order differential equations in which the amplitudes $\mu_{\mathbf{q}}$ are mixed only in 2×2 blocks. Letting $\mathbf{y}$ denote the vector $(\mu_{\mathbf{q}},\mu^*_{-\mathbf{q}})$, we obtain an equation of the form of Eq. (5) with

$$\hat{\mathbf{J}}=\begin{bmatrix}-[\alpha q^2+\beta a_{\mathbf{k}}^2] & -\beta a_{\mathbf{k}}^2\\ -\beta^* a_{\mathbf{k}}^2 & -(\alpha^{*2}+\beta^* a_k^2)\end{bmatrix}+2\mathbf{k}\cdot\mathbf{q}\begin{bmatrix}-\alpha & 0\\ 0 & \alpha^*\end{bmatrix}, \tag{22}$$

and

$$\hat{\mathbf{M}}=\begin{bmatrix}m & 0\\ 0 & m^*\end{bmatrix}, \tag{23}$$

with $\alpha\equiv 1+ic_1$ and $\beta\equiv 1-ic_3$.

Note that both $\hat{\mathbf{M}}$ and the first term in Eq. (22) are of the form $\hat{\mathbf{J}}_l$, but the term in Eq. (22) proportional to $\mathbf{k}\cdot\mathbf{q}$ is not. Thus the system describing the evolution of the complex vector $(\mu_{\mathbf{q}},\mu^*_{-\mathbf{q}})$ is a candidate for satisfying the conditions of the theorem if and only if $\mathbf{k}\cdot\mathbf{q}=0$.

In one dimension, we never have $\mathbf{k}\cdot\mathbf{q}=0$, so the theorem does not apply. Indeed, it is known that spatially local ETDAS control can extend the domain of parameters c_1 and c_3 over which a plane wave is stable [6]. In two or more dimensions, however, there always exist perturbation wave vectors for which $\mathbf{k}\cdot\mathbf{q}=0$. These must be analyzed further to determine whether the other conditions of the theorem are met.

Let $\hat{\mathbf{J}}_\perp$ indicate $\hat{\mathbf{J}}$ for the case $\mathbf{k}\cdot\mathbf{q}=0$. If $\mathrm{Det}[\hat{\mathbf{J}}_\perp]<0$, the theorem (Remark 3) will apply, meaning that there will be no choice of γ or m that suppresses the given transverse instability. Calculating the determinant from Eq. (22), we have $\mathrm{Det}[\hat{\mathbf{J}}_\perp]<0$ for all $q<q_{cr}$, where

$$q_{cr}^2=|a_{\mathbf{k}}|^2\frac{2(c_1c_3-1)}{1+c_1^2}. \tag{24}$$

Note that q_{cr} exists only in the regime $c_1c_3>1$. It turns out in this case that the criterion for transverse instability and the criteria for application of the theorem are identical. All unstable perturbation modes with $\mathbf{k}\cdot\mathbf{q}=0$ are immune to ETDAS.

For completeness, we note that ETDAS may work in the narrow parameter range where there exist unstable perturbations with $\mathbf{k}\cdot\mathbf{q}\neq 0$, but $c_1c_3<1$. Note also that in this region, although the plane wave under consideration may be unstable, there exist plane waves with smaller k that are stable even with no control.

IV. ADDITIONAL EXAMPLES

In this section we demonstrate the existence of unstable torsion-free perturbations to plane wave solutions of higher-order CGLE's in two or more dimensions. Symmetry considerations guarantee that transverse perturbations will be torsion-free, but a calculation is required to determine whether there is only a single unstable eigenvalue in the relevant subspace. For a broad family of CGLE's, unstable transverse perturbations occur within a band of wave numbers $0<q^2<q_c^2$ and ETDAS control always fails. As an example, we analyze the CGLE with a fifth-order term added. It is also shown that a $\mathbf{n}\cdot\boldsymbol{\nabla}$ term, which breaks the isotropy of the CGLE, does not help to extend the domain of controllable plane waves in more than two dimensions.

We consider systems of the form

$$\partial_t A=\epsilon A+\sum_{(i,j)\in(0,0)}^{L,J} a_{lj}|A|^{2l}\boldsymbol{\nabla}^{2j}A, \tag{25}$$

where a_{lj} are complex constants. We can find (for some cases) a traveling wave solution of the form

$$A_0(x,t)=\sqrt{M}e^{i(\mathbf{k}\cdot\mathbf{x}-\Omega_{\mathbf{k}}t)}. \tag{26}$$

The amplitude and frequency are determined by the equations

$$\epsilon=-\sum_{l,j}\ \mathrm{Re}[a_{lj}](-1)^j k^{2j}M^l \tag{27}$$

and

$$\Omega=\sum_{l,j}\ \mathrm{Im}[a_{lj}](-1)^j k^{2j}M^l. \tag{28}$$

Writing $A=A_k(1+B)$, expanding B in Fourier modes, and linearizing in the Fourier amplitudes, one obtains independent evolution equations for the pairs $(\mu_{\mathbf{q}},\mu^*_{-\mathbf{q}})$, with

$$\hat{\mathbf{J}}=\begin{bmatrix}\sum_{l,j} b_{lj}F_{l,2j}(\mathbf{k},\mathbf{q}) & \sum_{l,j} b_{lj}lk^{2j}\\ \sum_{l,j} b^*_{lj}lk^{2j} & \sum_{l,j} b^*_{lj}F_{l,2j}(\mathbf{k},-\mathbf{q})\end{bmatrix}, \tag{29}$$

where

$$b_{lj}\equiv(-1)^jM^la_{lj} \tag{30}$$

and

$$F_{l,2j}(\mathbf{k},\mathbf{q})=|k+q|^{2j}-k^{2j}+lk^{2j}. \tag{31}$$

Note that $F_{2j}(\mathbf{k},\mathbf{q})=F_{2j}(\mathbf{k},-\mathbf{q})$ if and only if $\mathbf{k}\cdot\mathbf{q}=0$.

For transverse perturbations $(\mathbf{k}\cdot\mathbf{q}=0)$, we have

$$\hat{\mathbf{J}}=\hat{\mathbf{J}}_\perp\equiv\begin{bmatrix}\rho+\sigma & \rho\\ \rho^* & \rho^*+\sigma^*\end{bmatrix}, \tag{32}$$

with $\rho\equiv\Sigma_{l,j}b_{lj}lk^{2j}$ and $\sigma\equiv\Sigma_{l,j}b_{lj}[(k^2+q^2)^j-k^{2j}]$. Note that $\hat{\mathbf{J}}_\perp$ has the form of $\hat{\mathbf{J}}_1$ (see Remark 3 above).

The fact that the transverse perturbations lack torsion follows from a combination of symmetry considerations. First, the symmetry of the CGLE with respect to spatial translations, $A(\mathbf{x},t)\to A(\mathbf{x}+\boldsymbol{\Delta},t)$, together with the symmetry under a global phase shift, $A(\mathbf{x},t)\to\exp(i\theta)A(\mathbf{x},t)$, guarantees the existence of simple, plane wave solutions for sufficiently small k. The linearized equations for B (defined as above) retain rotational symmetries about the $\mathbf{k}$ axis, reflection symmetries through planes containing that axis, and the translational symmetry of the original equations. Reflection symmetry through the plane perpendicular to $\mathbf{k}$ is broken, however, by a $\mathbf{k}\cdot\boldsymbol{\nabla}\mathbf{B}$ term. (Note that a variation in B at wave vector $\mathbf{q}$ corresponds to a variation in A at wave vector $\mathbf{k}+\mathbf{q}$.)

The translational symmetry governing the perturbations ensures that they can be resolved into sets of plane waves $B=\Sigma\mu(t)\exp(i\mathbf{q}\cdot\mathbf{x})$, with couplings only between waves with wave vectors $\mathbf{q}$ and $-\mathbf{q}$. For generic wave vectors, the two coupled waves are not related by any symmetry. For the special case of $\mathbf{k}\cdot\mathbf{q}=0$, however, the two waves in one sector are related by a reflection symmetry. Perturbations in this sector must either grow without translating in either the $\mathbf{q}$ or $-\mathbf{q}$ direction and hence have no torsion, or be resolvable into eigenmodes that are complex conjugate pairs. In other words, the form of $\hat{\mathbf{J}}_\perp$ is a consequence of two features of the linearized equation for B: the translational symmetry that permits couplings between wave vectors $\mathbf{q}$ and $-\mathbf{q}$ only; and the reflection symmetry that relates perturbations with those wave vectors in the case $\mathbf{k}\cdot\mathbf{q}=0$.

The systems under consideration are autonomous, and therefore possess a time-translation symmetry that guarantees that one of the eigenvalues in the $q=0$ sector must be identically zero. As shown in Ref. [8], the limitation on ETDAS control still applies in such a case; i.e., if the other $q=0$ mode is unstable, ETDAS will fail. If, however, the other $q=0$ mode is stable, then there are several possible situations. Let $\lambda_+(q^2)$ and $\lambda_-(q^2)$ be the eigenvalues of $\hat{\mathbf{J}}_\perp$, with either $\lambda_+>\lambda_-$ or $\lambda_+=\lambda_-^*$ for all q^2, and consider the case $\lambda_+(0)=0$, $\lambda_-(0)<0$. If $\lambda_+(0^+)>0$, then ETDAS control fails for small q^2 since $\lambda_-(0^+)$ is less than zero by continuity. In this case, there is a band of unstable, long wavelength, transverse perturbations that cannot be controlled by ETDAS. If $\lambda_+(0^+)<0$, the situation is more complicated. In particular, it is possible for the long wavelength modes to be stable, for the two eigenvalues to collide and become a complex conjugate pair for stable values of q^2, and then for their real parts to cross zero at some higher q^2. In such a case, the unstable transverse modes would not be subject to the no-control theorem and ETDAS control may be possible. This latter scenario requires that higher-order gradient terms be present ($a_{jl}\neq0$ for some $j\geq2$) and that a_{01} have a negative real part. A full analysis of this unusual case is beyond the scope of this work.

To see how the symmetry argument is reflected in the algebra of a specific case that is slightly more complicated than the cubic case treated in Sec. III consider the CGLE with a quintic term:

$$\partial_t A = \epsilon A + (1 + ic_1)\nabla^2 A - (b_3 - ic_3)|A|^2 A - (1 - ic_5)|A|^4 A. \tag{33}$$

Both ϵ and b_3 can in principle take on any real value. The real part of the coefficient of the quintic term can be fixed at unity without loss of generality (assuming the equation must not have any divergent solutions). In the notation of Eq. (25), we have

$$a_{01} = 1 + ic_1,$$

$$a_{10} = -(b_3 - ic_3), \quad a_{20} = -(1 - ic_5),$$

and all other $a_{lj} = 0$. A traveling wave solution of the form of Eq. (26) must have

$$M = \frac{1}{2}[-b_3 \pm \sqrt{b_3^2 + 4(\epsilon - k^2)}], \tag{34}$$

$$\Omega_k = c_1 k^2 - (c_3 M + c_5 M^2). \tag{35}$$

Note that the amplitude $\sqrt{M}$ must be real. Let M_+ and M_- indicate the solutions corresponding to the different choices of sign of the square root. For $b_3 > 0$, the M_+ solution is relevant for $k^2 < \epsilon$, but the M_- solution is unphysical. For $b_3 < 0$, both solutions are possible for some values of ϵ and k.

To see that ETDAS control cannot work, it is sufficient to consider the trace of $\hat{\mathbf{J}}_\perp$:

$$\mathrm{Tr}[\hat{\mathbf{J}}_\perp] = -[M_\pm(\pm\sqrt{b_3^2 + 4(\epsilon - k^2)}) + q^2]. \tag{36}$$

For all M_+ solutions, the square root is positive, so $\mathrm{Tr}[\hat{\mathbf{J}}_\perp]$ is negative for all q. Thus it is impossible for both eigenvalues to have a positive real part and the only way for an instability to arise is to have $\mathrm{Det}[\hat{\mathbf{J}}_\perp] < 0$, in which case the theorem applies. For the M_- solutions, the square root is negative, so the trace is positive for sufficiently small q. In this case it is sufficient to consider the $q = 0$ sector, where it is straightforward to confirm that one of the eigenvalues of $\hat{\mathbf{J}}_\perp$ is zero, and therefore the other eigenvalue must be positive. Again, the number of positive real eigenvalues is odd and the theorem applies.

The symmetry argument survives even when a term proportional to $\mathbf{n} \cdot \nabla A$ is present in the CGLE, explicitly breaking the rotational invariance. The reflection symmetry through the plane containing both $\mathbf{k}$ and $\mathbf{n}$ still guarantees torsion-free perturbative modes. A straightforward calculation confirms this, showing the Jacobian to be of the form

$$\hat{\mathbf{J}}_n = \begin{bmatrix} -[\alpha q^2 + \beta a_k^2] & -\beta a_k^2 \\ -\beta^* a_k^2 & -[\alpha^* q^2 + \beta^* a_k^2] \end{bmatrix} + 2\mathbf{k}\cdot\mathbf{q}\begin{bmatrix} -\alpha & 0 \\ 0 & \alpha^* \end{bmatrix} + \mathbf{n}\cdot\mathbf{q}\begin{bmatrix} \nu & 0 \\ 0 & -\nu^* \end{bmatrix}, \tag{37}$$

where α, β, and ν are combinations of the complex coefficients in the CGLE.

It is clear that the conditions of the theorem (Remark 3) will be satisfied when the sum of the terms linear in $\mathbf{q}$ vanish. This occurs for any $\mathbf{q} \propto \mathbf{k} \times \mathbf{n}$. Thus, plane waves in three or more dimensions that are unstable to transverse perturbations with wave vectors perpendicular to $\mathbf{n}$ cannot be stabilized using spatially local ETDAS. In two dimensions, stabilization may be possible so long as $\mathbf{k}$ and $\mathbf{n}$ are not collinear.

V. CONCLUSIONS

For the CGLE in one dimension, Bleich and Socolar [6] showed that the domain of stable plane waves can be enlarged significantly using spatially local ETDAS. Here we have demonstrated that in higher dimensions the same method does not work. The reason is an interesting one: for any parameter values such that transverse perturbations to the desired plane wave ($\mathbf{q} \cdot \mathbf{k} = 0$) are unstable, the dynamics of those with sufficiently small wave number produce no torsion in the relevant subspace of Fourier amplitudes [the $(\mu_{\mathbf{q}}, \mu^*_{-\mathbf{q}})$ subspace] and therefore cannot be suppressed.

We have analyzed only the most straightforward implementation of ETDAS in the CGLE. An important feature of the feedback term we chose is that it does not generate any coupling between the 2×2 blocks of Fourier amplitudes that arise in the standard stability analysis of the CGLE. We note that in some physical systems, the feedback may break the global phase shift symmetry of the CGLE, leading to additional couplings. This occurs, for example, when the feedback term explicitly treats the real and imaginary parts of A differently. The effect of breaking the global phase shift symmetry is to introduce (time-dependent) feedback couplings between the $\mathbf{q}$ and $\mathbf{q} + 2n\mathbf{k}$ perturbation sectors, for all integers n. This fails to circumvent the theorem, however, because the uncontrolled eigenvalues corresponding to the 2×2 block associated with $-n$ are precisely the complex conjugates of those associated with n, so the full Jacobian for any finite truncation of the ladder of coupled modes still has an odd number of torsion-free modes.

Pyragas has recently suggested a new method for stabilizing torsion-free orbits with time-delay feedback [10]. The idea is to introduce into the system an auxiliary variable that adds one unstable, torsion-free perturbation mode, thereby changing the total number of unstable torsion-free modes to an even number. We note that application of this method to the cases studied above encounters serious difficulties. We have not examined all possible variations on this theme, which might include Fourier filtering of the feedback signal or feedback that breaks the global phase shift symmetry. The most straightforward attempts to adapt Pyragas's scheme to plane waves in the CGLE fail, however, either because they introduce pairs of unstable eigenvalues in the relevant sectors or they add unstable torsion-free eigenvalues to sectors that were previously controllable.

The stabilization of spatiotemporal dynamics using time-delay feedback is of interest primarily because implementation of the controller does not require the construction of any external representation of the system. The stabilization of plane waves against transverse perturbations appears difficult to achieve using straightforward time-delay methods. Additional work is needed to determine whether such methods can be useful in more than one spatial dimension when the desired orbit has a more complex spatial or temporal structure.

ACKNOWLEDGMENTS

The authors gratefully acknowledge helpful conversations with E. Schoell, M. Silber, H. S. Greenside, and M. R. Plesser. This research was supported by NSF Grant No. PHY-98-70028.

[1] E. Ott, C. Grebogi, and J. A. Yorke, Phys. Rev. Lett. **64**, 1196 (1990).
[2] K. Pyragas, Phys. Lett. A **170**, 421 (1992).
[3] J. E. S. Socolar, D. W. Sukow, and D. J. Gauthier, Phys. Rev. E **50**, 3245 (1994).
[4] M. E. Bleich and J. E. S. Socolar, Phys. Lett. A **210**, 87 (1996).
[5] D. W. Sukow, M. E. Bleich, J. E. S. Socolar, and D. J. Gauthier, Chaos **7**, 560 (1997).
[6] M. E. Bleich and J. E. S. Socolar, Phys. Rev. E **54**, R17 (1996).
[7] H. Nakajima, Phys. Lett. A **232**, 207 (1997).
[8] H. Nakajima and Y. Ueda, Physica D **111**, 143 (1998).
[9] M. C. Cross and P. C. Hohenberg, Rev. Mod. Phys. **65**, 851 (1993).
[10] K. Pyragas, Phys. Rev. Lett. **86**, 2265 (2001).